GEOMETRIC GROUP THEORY - SEMESTRAL EXAM.

Time : 3 hours Max. marks : 60

Answer all questions. You may use results proved in class after correctly quoting them. Any other claim must be accompanied by a proof.

(1) Let G, H be the groups defined by the presentations

$$G \left\langle a, b \,/\, abab^{-1} \right\rangle, \quad H = \left\langle u, v \,/\, u^2v^2 \right\rangle.$$

(a) Prove that $G \cong H$.
(b) Show that G is an infinite group and the elements a, b are of infinite order.
(c) Show that G is non-abelian.
(d) Show that the center $Z(G) \neq \{1\}$.
(e) Show that the abelianization G_{ab} of G is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$.
(f) Show that G is torsion free.
(g) Exhibit H as a free product with amalgamation.
(h) Exhibit G as an HNN-extension.
(i) Is G virtually cyclic?
(j) Show that G does not contain a non-abelian free group. [$10 \times 3 = 30$]

(2) (a) When are two metric spaces quasi-isometric? Show that a map between metric spaces that is at a bounded distance from a quasi-isometry is itself a quasi-isometry. [2+5]
(b) Show from the definition that $[0, \infty)$ and $\mathbb{R}$ (with the usual metrics) are not quasi-isometric. [8]

(3) (a) Let G be a finitely generated group with a finite generating set X. Discuss the definition of the growth function of G with respect to X. Compute the growth functions of the groups $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ with respect to finite generating sets of your choice. Using this, or otherwise, show that $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ are not quasi-isometric. [1+4+2]
(b) Define the various (equivalent) notions of hyperbolicity of a geodesic metric space. Show that there does not exist a quasi-isometric embedding $f : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}_2 * \mathbb{Z}_2$. Is there a quasi-isometric embedding $g : \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$? [2+4+2]